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# PRODUCTS OF $k$ -SPACES, AND $k$ -NETWORKS (General and Geometric Topology)

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PRODUCTS OF  $k$ -SPACES, AND  $k$ -NETWORKS

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We give some affirmations and negations to the following *Hypothesis*.

We assume that spaces are regular and  $T_1$ .

*Hypothesis*: Let  $X$  and  $Y$  be  $k$ -spaces with point-countable  $k$ -networks.

Then,  $X \times Y$  is a  $k$ -space if and only if one of the following holds.

( $K_1$ )  $X$  and  $Y$  have point-countable bases.

( $K_2$ )  $X$  or  $Y$  is locally compact.

( $K_3$ )  $X$  and  $Y$  are locally  $k_\omega$ -spaces.

Note that the “if” part of *Hypothesis* holds for any  $k$ -spaces  $X$  and  $Y$ .

Lasnev spaces; quotient  $s$ -images of metric spaces; or CW-complexes are  $k$ -spaces with a point-countable  $k$ -network.

We recall some definitions used in this note. Let  $X$  be a space, and let  $\mathcal{C}$  be a cover of  $X$ . Then,

$\mathcal{C}$  is *point-countable* (resp. *compact-countable*; *star-countable*) if any point  $x \in X$  (resp. compact  $K \subset X$ ; element  $C \in \mathcal{C}$ ) meets at most countably many  $D \in \mathcal{C}$ .

$X$  is *determined by*  $\mathcal{C}$ , if  $F \subset X$  is closed in  $X$  iff  $F \cap C$  is closed in  $C$  for every  $C \in \mathcal{C}$ . Here, we can replace “closed” by “open”.

$X$  is *dominated by*  $\mathcal{C}$ , if for any subcollection  $\mathcal{C}^*$  of  $\mathcal{C}$ ,  $\bigcup \mathcal{C}^*$  is a closed subset determined by  $\mathcal{C}^*$ .

A space is a *k-space* if it is determined by a cover of compact subsets.

Every space is determined by any open cover. Every space is dominated by any HCP (=hereditarily closure-preserving) closed cover. Every CW-complex is a  $k$ -space dominated by a cover of compact metric subsets.

Let  $\alpha$  be an infinite cardinal. Then, a space is a  $k_\alpha$ -space if it is determined by a cover  $C$  of compact subsets with  $|C| \leq \alpha$ . A space  $X$  is *locally*  $<k_\alpha$  if each  $x \in X$  has a nbd whose closure is a  $k_{\alpha(x)}$ -space, here  $\alpha(x) < \alpha$ . A space  $X$  is *locally*  $k_\omega$  if  $X$  is locally  $<k_\alpha$ ,  $\alpha = \omega_1$ .

Let  $X$  be a space, and let  $\mathcal{P}$  be a cover of  $X$ . Then,

$\mathcal{P}$  is a  $k$ -network if, for any nbd  $U$  of a compact set  $K$  in  $X$ , there is a finite  $\mathcal{P}^* \subset \mathcal{P}$  such that  $K \subset U \subset \mathcal{P}^*$ .

$\mathcal{P}$  is a  $cs^*$ -network (resp.  $cs$ -network) if, for any open nbd  $U$  of  $x \in X$ , and for any sequence  $L$  converging to  $x$ , there is  $P \in \mathcal{P}$  with  $x \in P \subset U$ , and  $P$  contains  $L$  frequently (resp. eventually).

A  $k$ -network  $\mathcal{P}$  is *closed* if elements of  $\mathcal{P}$  are closed.

Obviously,  $cs$ -networks; or *closed*  $k$ -networks are  $cs^*$ -networks.

Spaces with a countable  $k$ -network (resp.  $\sigma$ -locally finite  $k$ -network) are  $\aleph_0$ -spaces (resp.  $\aleph$ -spaces).

Spaces with a countable network are *cosmic spaces*.

Let  ${}^\omega\omega$  be the set of all functions from  $\omega$  to  $\omega$ . For  $f, g \in {}^\omega\omega$ ,  $f \geq g$  if  $\{n \in \omega : f(n) < g(n)\}$  is finite. Let  $b = \min \{|A| : \exists \text{ unbounded } A \subset {}^\omega\omega\}$ .

Let  $BF(\omega_2)$  mean " $b \geq \omega_2$ ", and  $\neg BF(\omega_2)$  mean " $b = \omega_1$ ". Then,

$(CH) \Rightarrow \neg BF(\omega_2)$ , and  $(MA + \neg CH) \Rightarrow BF(\omega_2)$ .

Let  $X$  and  $Y$  be  $\aleph$ -spaces; or closed  $s$ -images of metric spaces. Then, Hypothesis holds ( $[T_1]$ ).

Let  $X$  and  $Y$  be Lašnev spaces. Then,  $(CH) \Rightarrow \textit{Hypothesis}$  holds  $([T_2])$ ; and,  $\neg BF(\omega_2) \Leftrightarrow \textit{Hypothesis}$  holds  $([G])$ .

Let  $X$  and  $Y$  be CW-complexes (or closed images of CW-complexes). Then,  $\neg BF(\omega_2) \Leftrightarrow \textit{Hypothesis}$  holds  $([T_3]; [TZ])$ .

Now, let us consider following properties.

- (C) Closed  $\sigma$ -compact, cosmic subsets are  $\aleph_0$ -spaces.
- (A<sub>1</sub>) Space with a  $\sigma$ -locally countable  $cs^*$ -network.
- (A<sub>2</sub>) Fréchet space with a point-countable  $cs^*$ -network.
- (A<sub>3</sub>) Space with a star-countable  $cs^*$ -network.
- (A<sub>4</sub>) Space with a point-countable  $cs$ -network.
- (A<sub>5</sub>) Space with a compact-countable  $cs^*$ -network.
- (B<sub>1</sub>) Fréchet space with a point-countable  $k$ -network.
- (B<sub>2</sub>) Space with a  $\sigma$ -HCP  $k$ -network.
- (B<sub>3</sub>) Space with a star-countable  $k$ -network.
- (B<sub>4</sub>) Space with a  $\sigma$ -compact-finite  $k$ -network.
- (B<sub>5</sub>) Space with a compact-countable  $k$ -network.

Then,  $(A_1) \text{ or } (B_1) \Rightarrow (C)$ ;  $(A_2) \Rightarrow (B_1)$ ;  $(B_2) \text{ or } (B_3) \Rightarrow (B_4)$ , etc.

We note that first countable spaces with  $(B_4)$  are metric, etc.

Let  $X$  be a space, and let  $\{L_\gamma: \gamma < \omega_1\}$  be a collection of disjoint sequences in  $X$  with  $L_\gamma \rightarrow x_\gamma \notin L_\gamma$ . Let  $S = \bigcup \{L_\gamma: \gamma < \omega_1\} \cup L$ , here  $L = \{x_\gamma: \gamma < \omega_1\}$ . For a compact metric subset  $K$  of  $X$  with  $K \cap S = L$ , let  $K^* = S \cup K$  be a subspace of  $X$  with  $K^*/K = (\sum L_\gamma^*)/L$ , where  $L_\gamma^* = L_\gamma \cup \{x_\gamma\}$ . Here,  $K^*/K$  is a quotient space obtained from  $K^*$  by identifying all points of  $K$  to a single point, and  $\sum L_\gamma^*$  is the topological sum of  $\{L_\gamma^*: \gamma < \omega_1\}$ .

By the  $X$  and  $Y$ , we mean the spaces  $X$  and  $Y$  in *Hypothesis*; that is, the  $X$  and  $Y$  are  $k$ -spaces with point-countable  $k$ -networks.

In the following theorem, (C) is essential by Theorem 3(3) below.

**Theorem 1.** Let the  $X$  and  $Y$  satisfy (C). Then,

(1) If neither  $X$  nor  $Y$  contains a closed copy of any  $K^*$ , *Hypothesis* is valid.

(2) *Hypothesis* is valid iff  $\neg \text{BF}(\omega_2)$ .

(3) If  $X=Y$ , *Hypothesis* is valid.

**Corollary.** Let the  $X$  satisfy one of  $(A_1)$ , and let the  $Y$  do so. Then, *Hypothesis* is valid.

[The result for  $(A_1)$ ;  $(A_2)$ ; and  $(A_4)$  is due to [S]; [S or LLi]; and [LiL] resp. The result for  $(A_5)$  is due to [LLi], where spaces have compact-countable *closed*  $k$ -networks]

**Corollary.** Let the  $X$  satisfy one of  $(B_1)$ , and let the  $Y$  do so. Then, *Hypothesis* is valid iff  $\neg \text{BF}(\omega_2)$ . If  $X=Y$ , *Hypothesis* is valid. [The result for  $(B_2)$ ;  $(B_3)$ ;  $(B_5)$  is due to [L]; [LT<sub>1</sub>]; [LT<sub>2</sub>] resp.]

**Corollary.** Let  $X$  and  $Y$  be  $k$ -spaces with point-countable *closed*  $k$ -networks such that  $\text{so}(X) \leq 2$ , and  $\text{so}(Y) \leq 2$ , here  $\text{so}(X)$  is the sequential order of  $X$  (see [AF]). Then, *Hypothesis* is valid iff  $\neg \text{BF}(\omega_2)$ . If  $X=Y$ , *Hypothesis* is valid.

[The result is due to [S] under (CH)]

**Theorem 2** (MA). Let the  $X$  satisfy one of  $(B_1)$ , and let the  $Y$  satisfy one of  $(A_1)$  or  $(B_1)$ . Then, *Hypothesis* is valid, but replace  $(K_3)$  by  $(K_3^*)$ : One of  $X$  and  $Y$  is locally  $k_\omega$ , and another is locally  $< k_c$ ,  $c=2^\omega$ .

Let us say that an operation is CD if it is the decomposition of operators "closed maps" and "dominations". Let CD(metric) be the class of all spaces obtained from metric spaces under CD. Then, for example, spaces dominated by  $\check{\text{Lasnev}}$  spaces belong to CD(metric).

Note that spaces in CD(metric) have  $\sigma$ -compact-finite  $k$ -networks, and spaces in CD(separable metric) have star-countable  $k$ -networks.

**Corollary.** Let  $X$  and  $Y$  be spaces in the class CD(metric). Then,

(1) Under  $\neg \text{BF}(\omega_2)$ ,  $X \times Y$  is a  $k$ -space iff one of the following holds.

( $k_1$ )  $X$  and  $Y$  are metric.

( $k_2$ )  $X$  or  $Y$  is locally compact metric.

( $k_3$ )  $X$  is dominated by a countable cover of locally compact metric spaces, and so is  $Y$ . If  $X=Y$ , the result holds without  $\neg \text{BF}(\omega_2)$ .

(2) Under (MA), the result in (1) holds, but replace ( $k_3$ ) by

( $k_3^*$ ) One of  $X$  and  $Y$  is dominated by a countable cover of locally compact metric spaces, and another is the topological sum of  $k_\alpha$ -spaces, here  $\alpha < 2^\omega$ .

(3)  $X^\omega$  is a  $k$ -space iff  $X$  is metric.

**Theorem 3** (1) Under  $\text{BF}(\omega_2)$ , Hypothesis is not valid for  $\check{\text{Lasnev}}$ , CW-complexes  $X$  and  $Y$  with star-countable  $k$ -networks.

(2) Under  $\text{BF}(\omega_2)$ , Hypothesis is not valid for  $k$ -spaces  $X$  and  $Y$  with point-countable *closed*  $k$ -networks (indeed,  $X$  and  $Y$  are quotient, finite-to-one images of locally compact metric spaces) ([LiL]).

(3) Under (CH), Hypothesis is not valid for  $\sigma$ -compact,  $k$ -spaces  $X$  and  $Y$ , where  $X$  is a  $\check{\text{Lasnev}}$ ,  $\aleph_0$ -space which is neither locally compact nor metric, and  $Y$  has a point-countable *closed*  $k$ -network with  $\text{so}(Y)=3$ , but  $Y$  is not locally  $k_\omega$  ( $X$  and  $Y$  are quotient  $s$ -images of locally compact metric spaces) ([S]).

*Question:* Let  $X$  and  $Y$  be  $k$ -spaces with point-countable  $k$ -networks.

- (1) If  $X^2$  is a  $k$ -space, then does  $X$  satisfy  $(K_1)$ , or  $(K_3)$  ?
- (2) As a characterization for  $X \times Y$  to be a  $k$ -space, are there different types of properties on  $X$ ,  $Y$  from  $(K_1)$ ,  $(K_2)$ ,  $(K_3)$ , and  $(K_3^*)$  ?

### References

- [AF] A. V. Arhanagel'skii and S. P. Franklin, Ordinal invariants for topological spaces, Michigan Math. Journal, 15(1968), 313-320.
- [G] G. Gruenhage,  $k$ -spaces and products of closed images of metric spaces, Proc. Amer. Math. Soc., 80(1980), 478-482.
- [LiL] S. Lin and C. Liu, On spaces with point-countable  $cs$ -networks, Topology and its Appl., 74(1996), 51-60.
- [L] C. Liu, Spaces with a  $\sigma$ -hereditarily closure-preserving  $k$ -network, Topology Proc., 18(1993), 179-188.
- [LLi] C. Liu and S. Lin,  $k$ -space property of product spaces, Acta. Math. Sinica. 13(1997), 537-544.
- [LT<sub>1</sub>] C. Liu and Y. Tanaka, Spaces with a star-countable  $k$ -network, and related results, Topology and its Appl., 74(1996), 25-38.
- [LT<sub>2</sub>] C. Liu and Y. Tanaka, Star-countable  $k$ -networks, compact countable  $k$ -networks, and related results, to appear in Houston J. Math.
- [S] A. Shibakov, Sequentiality of products of spaces with point-countable  $k$ -networks, Topology Proc., 20(1995), 251-270.
- [T<sub>1</sub>] Y. Tanaka, A characterization for the products of  $k$ - and  $\kappa_0$ -spaces and related results, Proc. Amer. Math. Soc., 59(1976), 149-156.
- [T<sub>2</sub>] Y. Tanaka, A characterization for the product of closed images of metric spaces to be a  $k$ -space, *ibid.*, 74(1979), 166-170.
- [T<sub>3</sub>] Y. Tanaka, Products of  $CW$ -complexes, *ibid.*, 86(1987), 503-507.
- [TZ] Y. Tanaka and Zhou Hao-xuan, Products of closed images of  $CW$ -complexes and  $k$ -spaces, Proc. Amer. Math. Soc., 92(1984), 465-469.